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Algebraic solution for the vector potential in the Dirac equation

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Abstract
The Dirac equation for an electron in an external electromagnetic field can be regarded as a singular set of linear equations for the vector potential. Radford's method of algebraically solving for the vector potential is reviewed, with attention to the additional constraints arising from non-maximality of the rank. The extension of the method to general spacetimes is illustrated by examples in diverse dimensions with both c- and a-number wavefunctions.

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1. Introduction

The Maxwell–Dirac equations are the coupled nonlinear partial differential equations which describe a classical electron interacting with an electromagnetic field. They are also the equations from which quantum electrodynamics is derived. Since the mathematical foundations of the latter remain unclear, the Maxwell–Dirac equations continue to be of interest [1–3]. Recently Radford [4] handled the Maxwell–Dirac equations firstly by solving the Dirac equation for the electromagnetic potential in terms of the wavefunction and its derivatives, and then substituting this solution in the Maxwell equations. This approach subsequently led to some physically interesting results [5–7] (for a review see [8]).

Despite the viability and potential importance of Radford's algebraic solution, at least for the treatment of the equations of classical electrodynamics for electrons and photons, it appears that the method has not appeared before in this context. One analysis which reached negative conclusions about the approach, and which may have engendered the lack of attention to it in the literature, is that of Eliezer [9]. In that paper, it was noted that the determinant of the matrix of coefficients of the vector potential A_μ in the Dirac equation actually vanishes, and that therefore a unique algebraic inversion was *not* possible. The aim of this paper is to reconcile [4] and [9], and to emphasize the legitimacy of the algebraic ansatz, despite the negative conclusions of [9], by a careful analysis of the nature of the Dirac equation regarded as a linear system [10] for A_μ. The main result is that the Dirac equation is indeed invertible if a *real* solution for the vector potential is required, and moreover that the treatment entrains an additional set of polynomial constraints on the wavefunction and its partial derivatives which must be carried forward in any further analysis. In section 2 below, the abstract formalism is developed, and (for the four-dimensional case) it is shown how the explicit manipulations, which rely on the structure of the Dirac algebra to derive the solution for the vector potential and the additional constraints, conform to the general setting (it is also pointed out that the solution can be regarded as including the mass, or more generally a Lorentz scalar potential, as a fifth unknown). This is done both in Lorentz-covariant Dirac spinor notation, and in van der Waerden 2-spinor notation. In section 3, the case of arbitrary (flat) spacetimes with signature (t, s) is taken up. Known results on the structure of the Dirac algebra (formally, the Clifford algebra $\mathcal{C}(t, s)$) in these cases are used to give an enumeration of constraints which are *quadratic* in the wavefunction and derivatives (in addition to current and partial axial current conservation, which hold in all cases). The four-dimensional results are recovered, and generalized to the case of a-number as well as c-number wavefunctions. A major outcome is a tabulation (table 2) of such constraints as to fermion wavefunction statistics and metric signature in diverse dimensions. Concluding remarks and prospects for further development of the work are given in section 4 below.

2. The four-dimensional Dirac equation

2.1. 8×4 *real system*

The Dirac equation for a fermion of charge q described by the spinor wavefunction ψ in the presence of an external electromagnetic potential may be written[3]

$$q\gamma^\mu \psi A_\mu = (\mathrm{i}\gamma^\mu \partial_\mu \psi - m\psi). \tag{1}$$

Following Eliezer [9], we write this as a matrix equation for A_μ;

$$M^\mu_\alpha A_\mu = Z_\alpha \tag{2}$$

where $M^\mu_\alpha \equiv \gamma^\mu{}_\alpha{}^\beta \psi_\beta$ for $M \in M_4(\mathbb{C})$, $M : \mathbb{C}^4 \longrightarrow \mathbb{C}^4$ and $A,\ Z \in \mathbb{C}^4$.

In [9] it was noted that M had rank 3 and determinant zero, with a rank 1 right null space, and therefore could not be inverted to obtain a unique solution for the potential. Yet Radford [4] did just this, albeit in the bispinor representation. That work exploited the fact that A_μ is real[4], which was not used in [9]. The point is that despite the zero determinant, (2) can be inverted if we know that A_μ is real, and providing that the intersection of the right null space of M with $\mathbb{R}^4$ (as a subspace of $\mathbb{C}^4$) is trivial. Even though the columns of M are not linearly independent as a vector space over $\mathbb{C}$, they are in general linearly independent as a vector space over $\mathbb{R}$.

We may break (2) into real and imaginary parts, yielding a system of eight real equations in four real unknowns, schematically $\mathcal{M}A = \mathcal{Z}$, where

$$\mathcal{M} = \begin{pmatrix} \frac{1}{2}(M + M^*) \\ \frac{1}{2\mathrm{i}}(M - M^*) \end{pmatrix} \qquad \mathcal{Z} = \begin{pmatrix} \frac{1}{2}(Z + Z^*) \\ \frac{1}{2\mathrm{i}}(Z - Z^*) \end{pmatrix}. \tag{3}$$

[3] In this section standard Cartesian coordinates x^μ, $\mu = 0, 1, 2, 3$ for four-dimensional Minkowski space with $(1, 3)$ metric $(\eta_{\mu\nu})_{\mu,\nu=0,1,2,3} = \mathrm{diag}(+, -, -, -)$ are introduced. Affices for Dirac spinors are introduced as ψ_α, $\alpha = 1, 2, 3, 4$, while the Dirac matrices (generators of the Clifford algebra $\mathcal{C}(1, 3)$ in the standard basis) satisfy $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$; for conventions see [11]. In section 3 below, the notation is generalized to dimension $d = t + s$ with metric signature (t, s).

[4] The most telling use of the reality of A_μ is implicit in the bispinor representation, where half the equations are conjugated, taking A_μ to be real. This results in systems of equations where the matrix operating on A_μ can have non-zero determinant [7]. This transformation cannot be performed with a matrix transformation on $\mathbb{C}^4$, and does not preserve the determinant.

$\mathcal{M}$ is not square; no determinant is defined; yet there are other tests for linear independence and invertibility [10]. To invert a system of m equations in n unknowns, with $m > n$, of the form (3), then we seek an $n \times m$ matrix $\mathcal{G}$ such that $\mathcal{G}\mathcal{M} = \mathbb{1}$, the unit $n \times n$ matrix, and then $A = \mathcal{G}\mathcal{Z}$. If such a solution exists, the rank of $\mathcal{M}$ is n $(= 4)$.

Note that the multiplication of a row of $\mathcal{G}$ with a column of $\mathcal{M}$ is actually a Euclidean real inner product. If the columns of $\mathcal{M}$ are understood to be spinors over an eight-dimensional real basis, we can accept the same interpretation for the rows of $\mathcal{G}$. The existence of an eight-dimensional real basis thus supplies us with a definition of a real inner product between spinors, for which we will use the notation $(\phi \cdot \psi)$. It is easy to verify that $(\phi \cdot \psi)$ is actually equal to the real part of the standard complex inner product:

$$(\phi \cdot \chi) = \mathrm{Re}\langle \phi, \chi\rangle = \tfrac{1}{2}(\phi^\dagger \chi + \psi^\dagger \chi). \tag{4}$$

The system of m equations in n unknowns entails [10] that the right-hand side of the equation should satisfy $m - n$ $(8 - 4 = 4)$ additional consistency conditions, arising from the fact that $\mathcal{Z}$ must fall in the column space of $\mathcal{M}$. To find these consistency conditions, we seek a further $m - n$ linearly independent spinor rows χ that have zero real inner product with the columns of $\mathcal{M}$ (χ span the left null space of $\mathcal{M}$). The consistency conditions may then be written $(\chi \cdot Z) = 0$. $\mathcal{G}$ is not unique, for any linear combination of the rows χ can be added without changing its effect on $\mathcal{M}$.

It is not necessary to work explicity in eight real components: regardless of which basis we use, the columns of $\mathcal{M}$ and the rows of $\mathcal{G}$ are just spinors in a vector space over Re, and the matrix multiplication is just the calculation of inner products using (4). All that we require for the inversion is to find spinors ϕ_ν where $(\phi_\nu \cdot \gamma^\mu \psi) \propto \delta^\mu_\nu$. For the consistency conditions, we require four linearly independent spinors χ such that $(\chi \cdot \gamma^\mu \psi) = 0$. As will now be shown, the structure of the Dirac algebra indeed admits such rows, ϕ_ν and χ.

2.2. Inversion

Let $\phi_\nu = \gamma^0 \gamma_\nu \psi$. Then

$$(\phi_\nu \cdot \gamma^\mu \psi) = \tfrac{1}{2}(\psi^\dagger \gamma_\nu^\dagger \gamma^{0\,\dagger} \gamma^\mu \psi + \psi^\dagger \gamma^{\mu\,\dagger} \gamma^0 \gamma_\nu \psi).$$

We use the hermiticity of γ^0 and $\gamma^0\gamma^\mu$:

$$\gamma^{\mu\dagger}\gamma^0 = \gamma^{\mu\dagger}\gamma^{0\dagger} = (\gamma^0\gamma^\mu)^\dagger = \gamma^0\gamma^\mu$$

and likewise for γ_ν. Then

$$(\phi_\nu \cdot \gamma^\mu \psi) = \tfrac{1}{2}(\psi^\dagger \gamma^0(\gamma_\nu\gamma^\mu + \gamma^\mu\gamma_\nu)\psi) = \delta_\nu{}^\mu \overline{\psi}\psi$$

where the Dirac conjugate $\overline{\psi}$ is defined in the usual way[5] as $\overline{\psi} = \psi^\dagger\gamma^0$.

Applying the real inner product with the same rows to the right-hand side of the Dirac equation (1), gives explicitly

$$\tfrac{1}{2}(\mathrm{i}\psi^\dagger\gamma^0\gamma^\nu\gamma^\mu\partial_\mu\psi - \mathrm{i}\partial_\mu\psi^\dagger\gamma^0\gamma^\mu\gamma^\nu\psi - 2m\psi^\dagger\gamma^0\gamma^\nu\psi).$$

The last term is identified with the current $j^\nu \equiv \overline{\psi}\gamma^\nu\psi$, so we can write the solution for the vector potential

$$A_\mu = \frac{1}{2q}\frac{\mathrm{i}(\overline{\psi}\gamma_\mu\partial\!\!\!/\psi - \overline{\psi}\overleftarrow{\partial\!\!\!/}\gamma_\mu\psi) - 2mj_\mu}{\overline{\psi}\psi}. \tag{5}$$

[5] General properties and nomenclature for the Dirac algebra in generic dimensions and spacetime signatures (including four-dimensional Minkowski space) are given in section 3 below.

2.3. Consistency conditions

It is possible to show by the use of (4) two separate sufficient conditions for spinors χ to have zero real inner product with $\gamma^\mu\psi$:

(1) $\chi = \Gamma\psi$, where Γ is a matrix in the Dirac algebra such that $\Gamma^\dagger\gamma^\mu$ is *anti*Hermitean; *Alternatively*,
(2) $\chi = \Upsilon\psi^*$ where Υ is a matrix in the Dirac algebra such that $\Upsilon^\dagger\gamma^\mu$ is *antisymmetric*.

As an example of (1), take $\chi = \mathrm{i}\gamma^0\psi$. Both the left-hand side as well as the mass term of (1) vanish (as $\mathrm{i}\gamma^0$ is itself antiHermitean), leaving

$$0 = -\psi^\dagger\gamma^0\gamma^\mu\partial_\mu\psi - \partial_\mu\psi^\dagger(\gamma^0\gamma^\mu)^\dagger\psi.$$

This is the normal current conservation equation,

$$\partial\cdot j \equiv \partial_\mu j^\mu = \partial_\mu\psi^\dagger\gamma^0\gamma^\mu\psi = 0. \tag{6}$$

Also satisfying condition (1), take $\chi = \mathrm{i}\gamma^0\gamma_5\psi$, using the antihermiticity of $\mathrm{i}\gamma^0\gamma_5\gamma^\mu$. The hermiticity of $\mathrm{i}\gamma_5\gamma^0$ now ensures that the mass term survives, and manipulations on the right-hand side of (1) lead in a similar way to the equation for partial conservation of axial current $j_5^\nu \equiv \overline{\psi}\gamma_5\gamma^\nu\psi$ as the second consistency condition:

$$\partial\cdot j_5 + 2\mathrm{i}m\overline{\psi}\gamma_5\psi = 0. \tag{7}$$

As an example of the sufficient condition (2), take $\chi = \gamma_5 C\psi^*$ and $\chi = \mathrm{i}\gamma_5 C\psi^*$ respectively, where C is the charge conjugation matrix. We evaluate these inner products with $\gamma^\mu\psi$ using the Hermitean conjugate $(\gamma_5 C\psi^*)^\dagger = \psi^t(\gamma_5 C)^\dagger = -\psi^t C\gamma_5$. These inner products are the real and imaginary parts of $-\psi^t C\gamma_5\gamma^\mu\psi$ respectively, which is zero by the antisymmetry of $C\gamma^5\gamma^\mu$. Applying the same row operation(s) to the right-hand side gives $0 = \psi^t C\gamma^5(\mathrm{i}\gamma^\mu\partial_\mu\psi - m\psi)$ or

$$\psi^t C\gamma^5\partial\!\!\!/\psi = 0 \tag{8}$$

(after using the antisymmetry of $C\gamma_5$ itself to eliminate the mass term) yielding one complex condition, or two real conditions on the spinor. This is the result previously reported by Eliezer [9] (who attributed to Dirac the antisymmetry argument using $C\gamma_5 = \alpha_x\alpha_z$ in the standard representation). The consistency conditions (6)–(8) are equivalent to Radford's [4] 'reality' conditions.

2.4. Alternative inversion

As mentioned above, the choice of nonsingular matrix inverting $\mathcal{M}$, and consequently the form of the final expression for A, is not unique. As an alternative choose $\phi_\nu = \mathrm{i}\gamma^0\gamma_5\gamma_\nu\psi$. We then find by a similar working to (2.2), using the anticommuting property of γ_5 with γ^μ, that

$$(\mathrm{i}\gamma^0\gamma_5\gamma_\nu\psi\cdot\gamma^\mu\psi) = -\delta^\mu_\nu\overline{\psi}\mathrm{i}\gamma_5\psi.$$

Applying the inner product with the same rows to the right-hand side of the Dirac equation (1), in this case the mass term vanishes, yielding an alternative solution for the vector potential:

$$A_\mu = \frac{\mathrm{i}}{2q}\frac{\overline{\psi}\gamma_5\gamma_\mu\partial\!\!\!/\psi - \overline{\psi}\gamma_5\overleftarrow{\partial\!\!\!/}\gamma_\mu\psi}{\overline{\psi}\gamma_5\psi}. \tag{9}$$

That (5) and (9) are indeed equivalent, and equivalent to [4], follows from the use of Fierz identities together with the auxiliary constraints (see section 3 below).

2.5. 8 × 5 *real system*

The inversion (9) does not contain any mass term. However, note that the pseudoscalar consistency condition (7) can be written

$$m = \frac{\mathrm{i}}{2}\frac{\overline{\psi}\gamma_5 \overleftarrow{\partial\!\!\!/}\, \psi + \overline{\psi}\gamma_5 \partial\!\!\!/ \psi}{\overline{\psi}\gamma_5\psi}. \tag{10}$$

The similarity between (9) and (10) suggests that the original system could have been considered as eight real equations in five unknowns, $qA_0, \ldots, qA_3$, and m (or more generally a Lorentz scalar potential). In this system, (9) and (10) provide an inversion, while (6) and the real and imaginary parts of (8) provide the three consistency conditions.

2.6. *2-spinor analysis*

Radford [4] and Booth and Radford [5] used van der Waerden notation in order to derive a complex form of the vector potential subject to additional reality conditions. Here the 2 spinor version is reached via the Weyl representation of the Dirac algebra (see e.g. [11]), wherein

$$\psi_\alpha = \begin{pmatrix} u_a \\ \bar{v}^{\dot{a}} \end{pmatrix} \qquad \psi^c_\alpha = \begin{pmatrix} v_a \\ \bar{u}^{\dot{a}} \end{pmatrix} \qquad \overline{\psi}^\alpha = -\begin{pmatrix} v^a \\ \bar{u}_{\dot{a}} \end{pmatrix}.$$

A generic matrix Γ in the Dirac algebra has matrix elements

$$\Gamma_\alpha{}^\beta = \begin{pmatrix} \Gamma_a{}^b & \Gamma_{a\dot{b}} \\ \Gamma^{\dot{a}b} & \Gamma^{\dot{a}}{}_{\dot{b}} \end{pmatrix}$$

in particular,

$$\gamma^\mu{}_\alpha{}^\beta = -\begin{pmatrix} 0 & \bar{\sigma}^\mu_{a\dot{b}} \\ \sigma^{\mu\dot{a}b} & 0 \end{pmatrix}.$$

where[6]

$$(\sigma^\mu)_{0\leqslant\mu\leqslant 3} = (\sigma^0, \boldsymbol{\sigma}) \qquad (\bar{\sigma}^\mu)_{0\leqslant\mu\leqslant 3} = (\sigma^0, -\boldsymbol{\sigma}). \tag{11}$$

The definitions (11) are consistent with

$$C_{\alpha\beta} = -\begin{pmatrix} \varepsilon_{ab} & 0 \\ 0 & \varepsilon^{\dot{a}\dot{b}} \end{pmatrix} \qquad C^{\alpha\beta} = -\begin{pmatrix} \varepsilon^{ab} & 0 \\ 0 & \varepsilon_{\dot{a}\dot{b}} \end{pmatrix}$$

together with $\varepsilon = \mathrm{i}\sigma^2$, that is, component-wise,

$$\varepsilon_{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \varepsilon_{\dot{a}\dot{b}} \qquad \varepsilon^{ab} = -\varepsilon_{ab} \qquad \varepsilon^{\dot{a}\dot{b}} = -\varepsilon_{\dot{a}\dot{b}}.$$

Starting then from

$$q\gamma^\mu A_\mu \psi = (\mathrm{i}\gamma^\mu \partial_\mu - m)\psi$$

and transcribing to 2-spinor form, the Dirac equation reads directly

$$q\bar{A}_{a\dot{b}}\bar{v}^{\dot{b}} = \mathrm{i}\bar{\partial}_{a\dot{b}}\bar{v}^{\dot{b}} + mu_a \qquad qA^{\dot{a}b}u_b = \mathrm{i}\partial^{\dot{a}b}u_b + m\bar{v}^{\dot{a}}$$

where[7]

$$\bar{A}_{a\dot{b}} \equiv \bar{\sigma}^\mu_{a\dot{b}} A_\mu \qquad A^{\dot{a}b} = \sigma_\mu{}^{\dot{a}b} A^\mu.$$

[6] The Pauli matrices are

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \qquad \text{and} \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

[7] Hermiticity, and raising and lowering of indices are entailed in the relations $\bar{\sigma}^\mu_{a\dot{a}} = \varepsilon_{ab}\varepsilon_{\dot{a}\dot{b}}\sigma^{\mu\dot{b}b}$, $(\sigma^{\mu\dot{a}b})^* = \sigma^{\mu\dot{b}a}$.

Finally taking complex conjugates,

$$\begin{aligned}
U:&\quad qA^{\dot{d}c}\bar{v}_{\dot{d}} = -\mathrm{i}\partial^{\dot{d}c}\bar{v}_{\dot{d}} + mu^c\\
\bar{V}:&\quad qA^{\dot{a}b}u_b = \mathrm{i}\partial^{\dot{a}b}u_b + m\bar{v}^{\dot{a}}\\
\bar{U}:&\quad qA^{\dot{c}d}v_d = \mathrm{i}\partial^{\dot{c}d}v_d + m\bar{u}^{\dot{c}}\\
V:&\quad qA^{\dot{b}a}\bar{u}_{\dot{b}} = -\mathrm{i}\partial^{\dot{b}a}\bar{u}_{\dot{b}} + mv^a.
\end{aligned}$$

Thus by taking combinations of the form $\alpha(\bar{V}^{\dot{a}}v^b - \bar{U}^{\dot{a}}u^b)$, $\beta(U^b\bar{u}^{\dot{a}} - V^b\bar{v}^{\dot{a}})$ and using $u_dv^b - v_du^b = \delta_d{}^b(u_cv^c)$ the vector potential can be isolated, with general 'solution'

$$A^{\dot{c}d} = -\frac{\mathrm{i}}{q}\left[\frac{\alpha(v^d\partial^{\dot{c}e}u_e + u^d\partial^{\dot{c}e}v_e) - \beta(\bar{u}^{\dot{c}}\partial^{\dot{e}d}\bar{v}_{\dot{e}} - \bar{v}^{\dot{c}}\partial^{\dot{e}d}\bar{u}_{\dot{e}}) - 2\mathrm{i}m(\alpha u^d\bar{u}^{\dot{c}} + \beta v^d\bar{v}^{\dot{c}})}{(\alpha u^a v_a + \beta\bar{u}^{\dot{a}}\bar{v}_{\dot{a}})}\right] \tag{12}$$

with arbitrary parameters α, β. As emphasized by Radford [4], all these forms are equivalent, subject to the hermiticity conditions satisfied by the potential itself. The latter can be imposed via the two-spinor projections of $A^{\dot{a}b}$, namely

$$\begin{aligned}
(A^{\dot{a}b}\bar{u}_{\dot{a}}u_b)^* &= (A^{\dot{a}b}\bar{u}_{\dot{a}}u_b)\\
(A^{\dot{a}b}\bar{v}_{\dot{a}}v_b)^* &= (A^{\dot{a}b}\bar{v}_{\dot{a}}v_b)\\
(A^{\dot{a}b}\bar{u}_{\dot{a}}v_b)^* &= (A^{\dot{a}b}\bar{v}_{\dot{a}}u_b).
\end{aligned} \tag{13}$$

Substitution of these conditions into a suitable form of (12), for example with $\alpha = 1,\ \beta = 0$, leads to

$$\begin{aligned}
&\partial_\mu(v^a\bar{\sigma}^\mu_{a\dot{b}}\bar{v}^{\dot{b}} + \bar{u}_{\dot{a}}\sigma^{\mu\dot{a}b}u_b) = 0\\
&\partial_\mu(v^a\bar{\sigma}^\mu_{a\dot{b}}\bar{v}^{\dot{b}} + \bar{u}^{\dot{a}}\sigma^{\mu\dot{a}b}u_b) = 2\mathrm{i}m(v^au_a - \bar{u}_{\dot{a}}\bar{v}^{\dot{a}})\\
&u^a\bar{\sigma}^\mu_{a\dot{b}}\partial_\mu\bar{v}^{\dot{b}} = \bar{v}_{\dot{a}}\sigma_\mu{}^{\dot{a}b}\partial^\mu u_b.
\end{aligned} \tag{14}$$

Referring (12) to a fixed tetrad basis, $\bar{u}_{\dot{a}}u_b$, $\bar{v}_{\dot{a}}v_b$, $\bar{u}_{\dot{a}}v_b$ and $\bar{v}_{\dot{a}}u_b$ via appropriate contractions and using (14) then shows directly that all forms are equivalent to the manifestly Hermitean version with $\alpha = \beta = 1$. As expected from the general analysis in the previous subsection, (12) with $\alpha = \beta = 1$ agrees with the previous result (5) expressed in the Weyl basis, and (14) are equivalent to current and partial axial current conservation[8] and the additional complex pseudoscalar identity (8) satisfied by the Dirac wavefunction.

3. Higher-dimensional extensions

In the previous section it was shown that the four-dimensional Dirac equation can be regarded as a singular set of real linear equations for the vector potential A_μ. Gaussian elimination in this 8×4 real system then leads to a solution for A_μ and also implies a set of four additional linearly independent constraints, linear in $\partial_\mu\psi$, which can be identified in this case with bilinear identities proposed by Radford [4] and Booth and Radford [5] using van der Waerden 2-spinor notation. Either approach ultimately derives from the structure of the Dirac algebra and the symmetry properties of the γ matrices. The same analysis is extended here in Dirac notation to higher-dimensional cases and different metric signatures (in flat spacetime). Also, an important distinction to make is that between c-number and a-number (or Grassmann-valued) Dirac spinors. For the latter, a Gaussian elimination argument requires a formal treatment of linear algebra over Grassmann-extended ground fields [12]; for present purposes it suffices to assume that the count of solutions and constraints goes through.

[8] Take the sum and difference of the first two lines of equation (14).

Table 1. The values of ϵ_B as a function of $\delta_B = \pm 1$ and $s - t$ (mod 8) (from [13]; $\times$ indicates that no representation with the specified signs exists).

$s-t$	0	1	2	3	4	5	6	7
ϵ_+	+1	×	−1	−1	−1	×	+1	+1
ϵ_-	+1	+1	+1	×	−1	−1	−1	×

The explicit construction of the Dirac algebra in higher dimensions has been reviewed by Tanii [13]; see also [14–16]. The Dirac matrices satisfy

$$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu} \tag{15}$$

where the spacetime metric signature is taken as (t, s) for even dimensions $t + s = d$, indices being labelled $\mu, \nu = 0, 0', 0'', \ldots; 1, 2, \ldots, s$, with chirality determined by the equivalent of γ_5, $\hat{\gamma} \equiv \gamma_0\gamma_{0'}\cdots\gamma_1\gamma_2\cdots\gamma_d$.

There are three involutive automorphisms associated respectively with complex conjugation, transposition and Hermitean conjugation under which the γ matrices acting on complex spinors ψ_α undergo similarity transformations[9],

$$\gamma_{\mu_\alpha}{}^{\beta} = \delta_A\ A_{\alpha\alpha'}\ \gamma_\mu^{\dagger\alpha'}{}_{\beta'}\ (A^{-1})^{\beta'\beta} \tag{16}$$

$$\gamma_{\mu_\alpha}{}^{\beta} = \delta_B\ B_\alpha{}^{\alpha'}\ \gamma^*_{\mu_{\alpha'}}{}^{\beta'}\ (B^{-1})_{\beta'}{}^{\beta} \tag{17}$$

$$\gamma_{\mu_\alpha}{}^{\beta} = \delta_C\ C_{\alpha\alpha'}\ \gamma_\mu^{T\alpha'}{}_{\beta'}\ (C^{-1})^{\beta'\beta} \tag{18}$$

where $\delta_A, \delta_B, \delta_C$ are sign factors depending on the spacetime:

$$\delta_A = \delta_B\delta_C$$
$$\delta_B = \pm 1$$
$$\delta_C = \delta_B(-1)^{t+1}.$$

A, B and C are related through the definitions of Dirac and charge conjugation for spinors,

$$\psi_{c\alpha} = B_\alpha{}^\beta \psi^*{}_\beta$$
$$\overline{\psi}^\alpha = \psi^*{}_\beta (A^{-1})^{\beta\alpha}$$
$$\psi^c{}_\alpha = C_{\alpha\beta}\overline{\psi}^\beta$$

so that

$$B_\alpha{}^\beta = C_{\alpha\beta'}(A^{-1})^{\beta\beta'}.$$

The fundamental identity $B^*B = \epsilon_B \mathbb{1}$ determines the existence of Majorana spinors for metrics in which $\epsilon_B = +1$, and $\delta_B = -1$. Dimensions for which this is possible can be read from table 1 which gives the values of $\epsilon_B = \sqrt{2}\cos[\frac{1}{4}\pi(s - t - \delta_B)]$. Finally, the implementation of parity in the Dirac algebra is determined by the symmetry of C, $C^T = \epsilon_C C$, $\epsilon_C = (\delta_B)^t(-1)^{\frac{1}{2}t(t-1)}\epsilon_B$.

Consider now the Dirac equation for ψ and conjugate forms:

$$\begin{aligned} q\not{A}\psi &= (\mathrm{i}\not{\partial} - m)\psi \\ q\not{A}\psi^c &= (-\mathrm{i}\not{\partial} - \delta_B m)\psi^c \\ q\overline{\psi}\not{A} &= \overline{\psi}(-\mathrm{i}\overleftarrow{\not{\partial}} - \delta_A m) \\ q\overline{\psi^c}\not{A} &= \overline{\psi^c}(\mathrm{i}\overleftarrow{\not{\partial}} - \delta_C m) \end{aligned} \tag{19}$$

[9] The notation $\gamma_\mu^{(a)}, \gamma_\mu^{(b)}, \gamma_\mu^{(c)}$ is used for the expressions on the right-hand sides of (18) (without the sign factors), extended to arbitrary elements of the Dirac algebra Γ (see below). Note that in the previous section the index conventions for the A and C matrices differ from that used here in the general case.

Table 2. Enumeration of the type of complex bilinear identity (scalar, S or pseudoscalar, P) admitted by the Dirac wavefunction of the indicated statistics, for various dimensions d and metrics of Minkowski $(1, d-1)$, conformal $(2, d-2)$ and Euclidean $(d, 0)$ signature (see text for details).

		$(1, d-1)$				$(2, d-2)$				$(d, 0)$			
		ε_B	S	P		$\delta_B\varepsilon_B$	S	P		$\pm\delta_B\varepsilon_B$	S	P	
	2	+1	(a)	(a)	(M)	—	—	—	—	−1	(c)	(c)	(M)
	4	∓1	(a)	(c)	(M)	±1	(c)	(a)	(M)	±1	a c	c a	—
d	6	−1	c	c	—	−1	(c)	(c)	(M)	+1	(a)	(a)	(pM)
	8	±1	(a)	(c)	(pM)	∓1	c a	a c	—	∓1	(a)	(c)	(M)
	10	+1	(a)	(a)	(M)	+1	(a)	(a)	(pM)	−1	(c)	(c)	(M)

where the sign factors follow from the explicit definitions in (18). Write (19) as Ψ, $\overline{\Psi}$, Ψ^c and $\overline{\Psi^c}$ respectively. Then, as in the four-dimensional case, the solution for the potential follows by taking combinations $\overline{\psi}\gamma_\mu\Psi + \overline{\Psi}\gamma_\mu\psi$ which force the isolation of A_μ through an anticommutator of γ matrices:

$$A_\mu = \frac{1}{2q}\frac{\mathrm{i}(\overline{\psi}\gamma_\mu\partial\!\!\!/\psi - \overline{\psi}\overleftarrow{\partial\!\!\!/}\gamma_\mu\psi) - 2s_A m\overline{\psi}\gamma_\mu\psi}{\overline{\psi}\psi} \tag{20}$$

where $s_A = \frac{1}{2}(1+\delta_A) \equiv \frac{1}{2}(1-(-1)^t)$ (compare (5)).

The identification of additional identities satisfied by the Dirac wavefunction given (19), (20) amounts to determining the structure of a certain ideal in the free algebra of rational expressions in ψ, ψ^* and partial derivatives $\partial^\mu\psi$, $\partial^\mu\psi^*$, for appropriately smooth wavefunctions. Such expressions do not necessarily form linear representations of the spacetime Lorentz symmetry group $SO(t,s)$, and the distinguished role played by the constraints following from Gaussian elimination is not clear. Here, the quadratic case is analysed, by analogy with the four-dimensional case, and in relation to the counting suggested by dimensional considerations. *Polynomials* in ψ, $\overline{\psi}$ and derivatives *do* decompose with respect to the Lorentz algebra. Moreover, there is a natural bi-grading by degree: in the quadratic case, this is simply by fermion number $\mathbb{F}$. Thus the cases $\overline{\psi}\cdot\psi$, with $\mathbb{F}=0$, and $\psi\cdot\psi$, or equivalently $\overline{\psi^c}\cdot\psi$, with $\mathbb{F}=2$, can be considered (as the latter will necessarily be complex, constraints in $\overline{\psi}\cdot\overline{\psi}$ do not require separate treatment). A case by case analysis follows.

$\mathbb{F} = \pm 2$

As is well known, products of two spinors admit a decomposition into antisymmetric tensor representations of the Lorentz group [17]. Noting that

$$\overline{\psi^c}\Gamma\psi \equiv \epsilon_B(C^{-1}\Gamma)^{\alpha\beta}\psi_\alpha\psi_\beta \tag{21}$$

it is clear that an explicit decomposition is provided by the linear basis for the Dirac algebra comprising the antisymmetric p-fold products of γ matrices [13, 18] $\{\gamma_\mu, \gamma_{\mu\nu}, \ldots, \gamma_{\mu_1\mu_2\ldots\mu_p}, \ldots\}$, $p = 1, \ldots, d$. Furthermore, for appropriate p, depending on the spacetime signature, dimension and statistics of the spinor wavefunctions, (21) is identically zero, leading after contraction with A_μ to a sequence of differential identities in the Dirac wavefunction, after implementation of (19), (20). If the statistics of the Dirac wavefunction is specified as $\psi_\alpha\psi_\beta = f\psi_\beta\psi_\alpha$, then using

$$\overline{\psi^c}\Gamma\psi = \epsilon_C f\overline{\psi^c}\Gamma^{(c)}\psi$$

and

$$(\gamma_{\mu\mu_1\mu_2\ldots\mu_p})^{(c)} = \delta_C^{p+1}\gamma_{\mu_p\ldots\mu_2\mu_1\mu} = \delta_C^{p+1}(-1)^{\frac{1}{2}(p+1)p}\gamma_{\mu\mu_1\mu_2\ldots\mu_p} \tag{22}$$

we require

$$\epsilon_C f \delta_C^{p+1}(-1)^{\frac{1}{2}(p+1)p} = -1. \tag{23}$$

Note also [18]

$$\begin{aligned}\gamma_\mu\gamma_{\mu_1\mu_2\ldots\mu_p} &= (-1)^p\gamma_{\mu_1\mu_2\ldots\mu_p}\gamma_\mu + 2p\eta_{\mu[\mu_1}\gamma_{\mu_2\ldots\mu_p]}\\ &= -(-1)^p\gamma_{\mu_1\mu_2\ldots\mu_p}\gamma_\mu + \frac{2}{(p+1)}\gamma_{\mu\mu_1\mu_2\ldots\mu_p}.\end{aligned} \tag{24}$$

Theorem 1. *For spacetime dimension $d = 4k$, a single Lorentz scalar or pseudoscalar bilinear complex differential constraint of the form*

$$\overline{\psi^c}\not{A}\psi = 0 \qquad or \qquad \overline{\psi^c}\hat{\gamma}\not{A}\psi = 0 \tag{25}$$

exists, whenever $\epsilon_C\delta_C f = \mp 1$, respectively. For spacetimes of dimension $d = 4k + 2$, both scalar and pseudoscalar complex constraints (25) hold if $\epsilon_C\delta_C f = -1$; otherwise neither holds. In general a sequence of Fierz-type identities

$$\delta_p\overline{\psi^c}\gamma_{\mu_1\mu_2\ldots\mu_p}\not{A}\psi = pA_{[\mu_1}\overline{\psi^c}\gamma_{\mu_2\ldots\mu_p]}\psi \tag{26}$$

holds, where $\delta_p = \frac{1}{2}(1-\epsilon_C\delta_C^{p+1}f(-1)^{\frac{1}{2}p(p+1)})$. Note that (25), (26) are regarded as differential conditions on ψ via the substitutions (19), (20).

Proof. Clearly (24), together with the interchange sign factors implies (26) after contraction with A_μ. After substituting (20) and rationalizing, the identity is therefore quartic, and generically of Fierz type. The only bilinear cases occur when $p = 0$ (in which case the condition $\epsilon_C\delta_C f = -1$ refers to the symmetry of $\overline{\psi^c}\not{A}\psi$), or for $p = d$ (for which γ_μ and $\hat{\gamma}$ anticommute, and the charge conjugation of $\hat{\gamma}$ determines whether the pseudoscalar identity holds). □

$\mathbb{F} = 0$

Theorem 2. *The real-type Fierz-type identities*

$$\overline{\psi}\not{A}\gamma_{\mu_1\mu_2\ldots\mu_p}\psi = (-1)^p\overline{\psi}\gamma_{\mu_1\mu_2\ldots\mu_p}\not{A}\psi + 2pA_{[\mu_1}\overline{\psi}\gamma_{\mu_2\ldots\mu_p]}\psi \tag{27}$$

hold, where the appropriate parts of (19) are to be used on the left- and right-hand sides, respectively. Current conservation and partial conservation of axial current hold for all d:

$$\begin{aligned}&\partial^\mu\overline{\psi}\gamma_\mu\psi = 0\\ &\partial^\mu\overline{\psi}\hat{\gamma}\gamma_\mu\psi + 2\mathrm{i}ms_A(\overline{\psi}\hat{\gamma}\psi) = 0.\end{aligned} \tag{28}$$

Proof. Equation (27) follows directly by contraction of (24a) with A_μ. Equation (28) uses (19) together with anticommutativity of γ_μ and $\hat{\gamma}$ (see the second line of equation (24)). □

Explicit enumeration of these results is complicated because of the multitude of subcases involved. In table 2 the counting and nature of the identities is illustrated in diverse dimensions, and for given fermion statistics and metric signatures for the Minkowski, conformal and Euclidean spacetimes. The explicit expressions for the sign factors in the representations of the Dirac algebra available imply the expressions given for $\delta_C\epsilon_C$, namely ε_B, $\delta_B\varepsilon_B$ and $(-1)^{\frac{1}{2}d}\delta_B\varepsilon_B$ for $t = 1, 2$ and d respectively. The entries within each metric signature class then indicate for which type of fermion wavefunction statistics (c-number, $f = +1$, or a-number, $f = -1$ respectively) the indicated scalar S or pseudoscalar P complex identity exists. Where $\varepsilon_B = +1$ exists, bracketed entries indicate the choice $\delta_B = -1$ or $\delta_B = +1$, consistent with the availability either of Majorana (M) or pseudoMajorana (pM) spinors respectively, in that dimension and spacetime signature.

In table 2, the entry for c-number wavefunctions in four-dimensional Minkowski space corresponding to a single complex bilinear pseudoscalar identity is the original Dirac equation case. For a-number fermions, there exists the corresponding scalar equivalent (reflecting the properties of γ_μ and $\gamma_\mu\gamma_5$ in the Dirac algebra). Taking into account the two real constraints (current and partial axial current conservation), there is in either case a total of four real conditions, in accord with the count needed to accompany the four-dimensional vector potential in the 8×4 linear system (see section 2).

The counting of constraints in d dimensions must be examined carefully in relation to Gaussian elimination in the corresponding $2^{\frac{1}{2}d+1} \times d$ real linear system. For example in two-dimensional Minkowski space there are *no* complex ($\mathbb{F} = \pm 2$) bilinear constraints for c-number wavefunctions, and thus only two real ($\mathbb{F} = 0$) bilinear constraints, whereas for a-number wavefunctions there are two real plus *two* complex identities. The apparent under- or over-determination of the system (at the quadratic level) must be reconciled with the remaining Fierz-type identities (of quartic type). Two dimensions is a special case because of the abelian nature of the Lorentz group, but in other cases, the Lorentz decomposition of the remaining higher-order constraints also bears on the counting. For example, in $d = 6$ (16×6 real system) there are again either no or two complex bilinear constraints (see table 2). In the latter case, a further $16 - 6 - 2 - 4 = 4$ independent real conditions are needed. These may represent four additional scalar conditions at higher order, or a real 6-vector which satisfies two additional conditions equivalent to two of the scalar and pseudoscalar conditions. Similar considerations apply to the $d = 8$ and 10 cases.

4. Conclusions

In this paper the linearity of the Dirac equation has been exploited as a vehicle to obtain an algebraic solution for the vector potential, and previous discussions in the literature [4] and [9] have been reconciled. In addition to its role in the problem of obtaining (classical) solutions of the full nonlinear Maxwell–Dirac equations, the algebraic method has potential application to the nonabelian case, and to Duffin–Kemmer algebras rather than Clifford algebras. Modifications to electrodynamics such as the Born–Infeld theory, and indeed the nonrelativistic limit of the Dirac equation itself, may also be amenable to further study by the method.

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